EMBEDDING q-DEFORMED HEISENBERG ALGEBRAS INTO UNDEFORMED ONES [1]

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Abstract

Any deformation of a Weyl or Clifford algebra can be realized through some change of generators in the undeformed algebra. Here we briefly describe and motivate our systematic procedure for constructing all such changes of generators for those particular deformations where the original algebra is covariant under some Lie group and the deformed algebra is covariant under the corresponding quantum group.

1. INTRODUCTION

Weyl and Clifford algebras are at the hearth of quantum physics. One may ask if deforming them, i.e. deforming their defining commutation relations, yields new physics [1], or at least may be useful to better describe some peculiar systems in conventional quantum physics. This question can be divided into an algebraic and a representation-theoretic subquestions. Roughly speaking, the first is: is there a formal realization of the elements of the deformed algebra in terms of elements of the undeformed algebra? The answer is affirmative [2, 3, 4] but in general the realization is not explicitly known. The second subquestion is: do also the corresponding representation theories coincide? One can already see in some simple model that the answer is

[1] Talk given at the XVI-th Workshop On Geometric Methods in Physics, July 1997, Bialowieza (Poland). To appear in the Proceedings, to be published on Rep. Math. Phys.

negative, but in the general case, up to our knowledge, the relation between the two is an open question.

We introduce the notions of a deformed algebra and of a deforming map first on a simplest toy model, the 1-dim Weyl algebra $\mathcal{A}$. $\mathcal{A}$ is generated by $\mathbf{1}, a, a^+$ fulfilling

$$a\, a^+ = \mathbf{1} + a^+ a \qquad\qquad \mathbf{1}b = b\mathbf{1} = b, \tag{1.1}$$

$b \in \mathcal{A}$. As a *deformation* $\mathcal{A}_h$ of $\mathcal{A}$ (h is the 'deformation parameter') we consider the algebra generated by $\mathbf{1}_h, \tilde{A}, \tilde{A}^+$ fulfilling the relations

$$\tilde{A}\, \tilde{A}^+ = \mathbf{1}_h + e^h \tilde{A}^+ \tilde{A} \qquad\qquad \mathbf{1}_h B = B\mathbf{1}_h = B, \tag{1.2}$$

$B \in \mathcal{A}_h$; when $h \to 0$ the second relations go to the first if we identify in the limit $\mathbf{1}_h, \tilde{A}, \tilde{A}^+$ with $\mathbf{1}, a, a^+$.

Can we realize $\mathbf{1}_h, \tilde{A}, \tilde{A}^+$ within $\mathcal{A}[[h]]$ (the ring of formal power series in the unknown h and with coefficients in $\mathcal{A}$), in other words as 'functions' of h, a, a^+ reducing to $\mathbf{1}, a, a^+$ in the limit? Yes. Let $n := a^+ a$, $q := e^h$, $(x)_q := \frac{q^x - 1}{q-1}$; if we define [5]

$$A := a\sqrt{\frac{(n)_q}{n}} \qquad\qquad A^+ := \sqrt{\frac{(n)_q}{n}}\, a^+, \tag{1.3}$$

it is easy to show that $\mathbf{1}, A, A^+$ indeed fulfil the 'deformed commutation relations' (DCR) (1.2): in other words $\mathbf{1}, A, A^+$ realize $\mathbf{1}_h, \tilde{A}, \tilde{A}^+$. At lowest order in h one finds $A = a + O(h)$, $A^+ = a^+ + O(h)$, as required. By definition, a *deforming map* f is an algebra isomorphism $f : \mathcal{A}_h \to \mathcal{A}[[h]]$ over $\mathbf{C}[[h]]$ reducing to the identity in the limit $h = 0$. We can obtain one by setting $f(\tilde{A}) := A$, $f(\tilde{A}^+) := A^+$ and extending its action on the whole $\mathcal{A}_h$ imposing $f(\alpha\beta) = f(\alpha)f(\beta)$ and linearity.

Here we shall deal with a particular class of deformations of multidimensional Weyl algebras or Clifford algebras (their fermionic counterparts). The undeformed algebra is covariant under some Lie algebra $\mathbf{g}$ and the deformed one under the quantum group [6] $U_h\mathbf{g}$. The undeformed algebra $\mathcal{A}$ is generated by $\mathbf{1}, a^i, a^+_j$ fulfilling

$$\begin{aligned} &[a^i\,,\, a^j]_\mp = 0 \\ &[a^+_i\,,\, a^+_j]_\mp = 0 \\ &[a^i\,,\, a^+_j]_\mp = \delta^i_j \mathbf{1} \end{aligned} \tag{1.4}$$

(the $\mp$ sign denotes commutators and anticommutators and refers to Weyl and Clifford algebras respectively) and transforms under the action $\triangleright$ of $\mathbf{g}$ according to some law

$$x \triangleright a^+_i = \rho(x)^j_i a^+_j \qquad\qquad x \triangleright a^i = \rho(Sx)^i_j a^j; \tag{1.5}$$

here $x \in \mathbf{g}$, $Sx = -x$ and ρ denotes some matrix representation of $\mathbf{g}$. Clearly a^i belong to a represenation of $\mathbf{g}$ which is the contragradient of the a^+_i one. The action $\triangleright$ is extended to products of the generators using the standard rules of tensor product representations (technically speaking, using the coproduct Δ of the universal enveloping algebra $U\mathbf{g}$), and then linearly to all of $\mathcal{A}$; this is possible because the action of $\mathbf{g}$ is manifestly compatible with the commutation relations (1.4). The same formulae, where S now denotes the antipode of $U\mathbf{g}$, give also the standard extension of $\triangleright$ to $x \in U\mathbf{g}$.

The corresponding deformed algebra $\mathcal{A}_h$ is generated by $\mathbf{1}_h, \tilde{A}^+_i, \tilde{A}^i$ fulfilling DCR which, in the simplest case of ρ being the defining fundamental

representation of $\mathbf{g}$, take the form [11, 12, 13]

$$\begin{aligned} &\mathcal{P}_{\mp}{}^{ij}_{hk}\tilde{A}^k\tilde{A}^h = 0 \\ &\mathcal{P}_{\mp}{}^{hk}_{ij}\tilde{A}^+_h\tilde{A}^+_k = 0 \\ &\tilde{A}^i\tilde{A}^+_j = \delta^i_j\,\mathbf{1} \pm q\hat{R}^{ih}_{jk}\tilde{A}^+_h\tilde{A}^k; \end{aligned} \tag{1.6}$$

$\mathcal{A}_h$ transforms under the action $\triangleright_h$ of $U_h\mathbf{g}$ according to the law [2]

$$x \triangleright_h \tilde{A}^+_i = \rho_h{}^j_i(x)A^+_j \qquad\qquad x \triangleright_h \tilde{A}^i = \rho_h{}^i_j(S_hx)A^j. \tag{1.7}$$

Here $x \in U_h\mathbf{g}$, S_h is the antipode of $U_h\mathbf{g}$, ρ_h the quantum group deformation of ρ, $\hat{R}$ the braid matrix of $U_h\mathbf{g}$ in the representation ρ_h, and finally $\mathcal{P}_-,\mathcal{P}_+$ are the $U_h\mathbf{g}$-covariant deformations of the antisymmetric and symmetric projectors, in the form of polynomials in $\hat{R}$; for instance, when $\mathbf{g} = sl(N)$ $\mathcal{P}_{\mp} = (q+q^{-1})^{-1}(q^{\pm 1}\mathbf{1} \mp \hat{R})$. The upper and lower sign refer to Weyl and Clifford algebras respectively. $\tilde{A}^i$ belong to a representation of $U_h\mathbf{g}$ which is the quantum group contragradient of the $\tilde{A}^+_i$ one. The action $\triangleright_h$ is extended to products of the generators using the coproduct Δ_h of $U_h\mathbf{g}$, see eq. (2.6) below, and then linearly to all of $\mathcal{A}_h$; this is possible because the action of $U_h\mathbf{g}$ is compatible with the commutation relations (1.6).

Is there a realization of $\mathbf{1}_h, \tilde{A}^i, \tilde{A}^+_j$ within $\mathcal{A}[[h]]$, or, in other words, a deforming map $f : \mathcal{A}_h \to \mathcal{A}[[h]]$? Yes. The affirmative answer is based on the vanishing of the second Hochschild cohomology group of any Weyl algebra [2, 3, 4]; this allows to prove the existence of f without however providing an explicit construction. The argument is valid not only for deformations of the type (1.6), but for *any* kind of deformation of $\mathcal{A}$.

In [8] we have suggested a systematic and explicit constructing procedure of deforming maps for the class of Weyl and Clifford algebras described above; the procedure is based on $U_h\mathbf{g}$-covariance and the socalled Drinfel'd twist [10]. In the sequel we briefly describe it. We shall motivate our physical interest in these deforming maps in the last section.

2. THE CONSTRUCTING PROCEDURE

If $\alpha \in \mathcal{A}[[h]]$ is any element of the form $\alpha = \mathbf{1} + O(h)$ and f is a deforming map, one can obtain a new one f_α by the inner automorhism

$$f_\alpha(\cdot) := \alpha f(\cdot)\alpha^{-1}; \tag{2.1}$$

actually the vanishing of the first Hochschild cohomology group of $\mathcal{A}$ implies that *all* deforming maps can be obtained from one in this manner. Therefore our problem is reduced to finding a particular one, what we are going to describe below.

The other essential ingredients of our construction procedure are:

1. $\mathbf{g}$, a simple Lie algebra.
2. The cocommutative Hopf algebra $H \equiv (U\mathbf{g}, \cdot, \Delta, \varepsilon, S)$ associated to $U\mathbf{g}$; $\cdot, \Delta, \varepsilon, S$ denote the product, coproduct, counit, antipode. We shall use the Sweedler's notation $\Delta(x) \equiv x_{(1)} \otimes x_{(2)}$: at the rhs a sum $\sum_i x^i_{(1)} \otimes x^i_{(2)}$ of many terms is implicitly understood.

[2] These $\mathcal{A}_h$ should not be confused with the celebrated Biedenharn-Macfarlane-Hayashi-Kulish q-oscillator (super)algebrae [7], whose generators α^i, α^+_j fulfil ordinary (anti)commutation relations, except for the q-(anti)commutation relations $\alpha^i\alpha^+_i \mp q^2\,\alpha^+_i\alpha^i = 1$, and are *not* $U_h\mathbf{g}$-covariant (in spite of the fact that they are usually used to construct a generalized Jordan-Schwinger realization of $U_h\mathbf{g}$).

3. The quantum group [6] $H_h \equiv (U_h\mathbf{g}\,, \bullet, \Delta_h, \varepsilon_h, S_h, \mathcal{R})$. $\bullet, \Delta_h, \varepsilon_h, S_h$ denote the deformed product, coproduct, counit, antipode, $\mathcal{R}$ the quasi-triangular structure. We shall use the Sweedler's notation (with barred indices) $\Delta_h(x) \equiv x_{(\bar{1})} \otimes x_{(\bar{2})}$.

4. An algebra isomorphism[10] $\varphi_h : U_h\mathbf{g} \;\rightarrow U\mathbf{g}\,[[h]]$ over $\mathbf{C}[[h]]$, namely $\varphi_h(x \bullet y) = \varphi_h(x) \cdot \varphi_h(y)$.

5. A corresponding Drinfel'd twist[10] $\mathcal{F} \equiv \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} = \mathbf{1}^{\otimes^2} + O(h) \in U\mathbf{g}[[h]]^{\otimes^2}$:

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)\mathcal{F}, \qquad \Delta_h(a) = (\varphi_h^{-1} \otimes \varphi_h^{-1})\{\mathcal{F}\Delta[\varphi_h(a)]\mathcal{F}^{-1}\}; \tag{2.2}$$

the last formula means that, up to the isomorphism φ_h, Δ_h is related to Δ by a similarity transformation.

6. $\gamma' := \mathcal{F}^{(2)} \cdot S\mathcal{F}^{(1)}$ and $\gamma := S\mathcal{F}^{-1(1)} \cdot \mathcal{F}^{-1(2)}$. Up to the isomorphism φ_h, S_h and its inverse are related to S by similarity transformations involving resp. γ and γ'.

7. The Jordan-Schwinger algebra homomorphism $\sigma : U\mathbf{g}\,[[h]] \rightarrow \mathcal{A}[[h]]$, defined on the generators by

$$\sigma(\mathbf{1}_{U\mathbf{g}}) = \mathbf{1} \qquad\qquad \sigma(x) := \rho(x)^i_j a^+_i a^j \tag{2.3}$$

$x \in \mathbf{g}$, and extended to the whole $U\mathbf{g}\,[[h]]$ as an algebra homomorphism, $\sigma(yz) = \sigma(y)\sigma(z)$ and $\sigma(y+z) = \sigma(y)+\sigma(z)$. This is consistent because $\sigma([x,y]) = [\sigma(x), \sigma(y)]$. In the $su(2)$ σ takes the well-known form

$$\sigma(j_+) = a^+_\uparrow a^\downarrow, \qquad \sigma(j_-) = a^+_\downarrow a^\uparrow, \qquad \sigma(j_0) = \frac{1}{2}(a^+_\uparrow a^\uparrow - a^+_\downarrow a^\downarrow). \tag{2.4}$$

8. The deformed Jordan-Schwinger algebra homomorphism $\sigma_h : U_h\mathbf{g} \;\rightarrow \mathcal{A}[[h]]$, defined by $\sigma_h := \sigma \circ \varphi_h$.

9. The $*$-structures $*, *_h, \star, \star_h$ in $H, H_h, \mathcal{A}, \mathcal{A}_h$, if $\mathcal{A}, \mathcal{A}_h$ are $*$-algebras transforming respectively under the Hopf $*$-algebras H, H_h with the compatibility condition

$$(x \triangleright_h a)^{\star_h} = S_h^{-1}(x^{*_h}) \triangleright_h a^{\star_h}. \tag{2.5}$$

2.1. Constructing the Quantum Group Action and the Generators A^i, A^+_j

Since we know that a deforming map exists, although we cannot write it explicitly we can say that it must be possible to realize $\triangleright_h$ on $\mathcal{A}[[h]]$, instead of $\mathcal{A}_h$. Our first step is to guess such a realization. This requires fulfilling

$$(xy) \triangleright_h a = x \triangleright_h (y \triangleright_h a) \qquad\qquad x \triangleright_h (ab) = (x_{(\bar{1})} \triangleright_h a)(x_{(\bar{2})} \triangleright_h b) \tag{2.6}$$

for any $x, y \in U_h\mathbf{g}\,,\ a, b \in \mathcal{A}_h$; these are the conditions characterizing a module algebra. There is a simple way to find such a realization, namely by setting

$$x \triangleright_h a := \sigma_h(x_{(\bar{1})}) a \sigma_h(S_h x_{(\bar{2})}); \tag{2.7}$$

it is easy to check that (2.6) are indeed fulfilled using the basic axioms characterizing the coproduct, counit, antipode in a generic Hopf algebra. The guess has been suggested by the cocommutative case, where the same conditions

and realization are obtained for $U\mathbf{g}\,,\mathcal{A},\triangleright$ if in the two previous formulae we just erase the suffix $_h$ and replace $\Delta_h(x)\equiv x_{(\bar{1})}\otimes x_{(\bar{2})}$ with the cocommutative coproduct $\Delta(x)\equiv x_{(1)}\otimes x_{(2)}$.

Our second step is to realize elements $A^i,A^+_j\in\mathcal{A}[[h]]$ that transform under (2.7) as $\tilde{A}^i,\tilde{A}^+_j$ in (1.7). Note that a^i,a^+_j do *not* transform in this way. In Ref. [14] we proved that the following objects do:

$$\begin{array}{lcl} A^+_i & := & u\,\sigma(\mathcal{F}^{(1)})a^+_i\sigma(S\mathcal{F}^{(2)}\gamma)\,u^{-1} \\ A^i & := & v\,\sigma(\gamma' S\mathcal{F}^{-1(2)})a^i\sigma(\mathcal{F}^{-1(1)})v^{-1}; \end{array} \tag{2.8}$$

the result holds for any choice of g-invariant elements $u,v=\mathbf{1}+O(h)$ in $\mathcal{A}[[h]]$, in particular for $u=v=\mathbf{1}$.

The third step is to fix u,v in such a way that the DCR are fulfilled. One can easily show that the DCR may fix at most the product uv^{-1}. For the explicit case considered in (1.6) we proved in Ref. [8] that the DCR are indeed fulfilled by taking uv^{-1}

$$uv^{-1} = \frac{\Gamma(n+1)}{\Gamma_{q^2}(n+1)} \tag{2.9}$$

$$uv^{-1} = \left(\frac{1+q^{N-2}}{2}\right)^{-n}\frac{\Gamma\left[\frac{1}{2}\left(n+\frac{N}{2}+1-l\right)\right]\,\Gamma\left[\frac{1}{2}\left(n+\frac{N}{2}+1+l\right)\right]}{\Gamma_{q^2}\left[\frac{1}{2}\left(n+\frac{N}{2}+1-l\right)\right]\,\Gamma_{q^2}\left[\frac{1}{2}\left(n+\frac{N}{2}+1+l\right)\right]} \tag{2.10}$$

respectively if $\mathbf{g}=sl(N),so(N)$. Here Γ is Euler's γ-function, Γ_{q^2} its q-deformation characterized by $\Gamma_{q^2}(x+1)=(x)_{q^2}\Gamma_{q^2}(x)$, $n:=a^ia^+_i$, and in the $\mathbf{g}=so(N)$ case we have enlarged for convenience $\mathcal{A}[[h]]$ by the introduction of the square root $l:=\sqrt{\sigma(\mathcal{C})}$, $\mathcal{C}$ being the quadratic casimir of $Uso(N)$. We stress that the above solutions regard the case of ρ being the defining representation of $\mathbf{g}$. We have yet no formula yielding the right uv^{-1}, if any, necessary to fulfil the DCR in the general case. However it is important to note that in general the DCR translate into conditions on uv^{-1} where the Drinfel'd twist $\mathcal{F}$ appears only through the socalled 'coassociator'

$$\phi:=[(\Delta\otimes\mathrm{id})(\mathcal{F}^{-1})](\mathcal{F}^{-1}\otimes\mathbf{1})(\mathbf{1}\otimes\mathcal{F})[(\mathrm{id}\otimes\Delta)(\mathcal{F}). \tag{2.11}$$

ϕ is explicitly known, unlike $\mathcal{F}$, for which up to now there is an existence proof but no explicit expression. This makes the above conditions explicit and allows to search the explicit form of uv^{-1} in the general case, if it exists.

Finally, the residual freedom in the choice of u,v is partially fixed if $H,H_h,\mathcal{A},\mathcal{A}_h$ are matched (Hopf) $*$-algebras and make the additional requirement that $\star$ realizes in $\mathcal{A}[[h]]$ the $\star_h$ of $\mathcal{A}_h$. For instance, if $(a^i)^\star=a^+_i$ and $h\in\mathbf{R}$ this means

$$(A^i)^\star=A^+_i, \tag{2.12}$$

and is fulfilled if we take $u=v^{-1}$.

In the $\mathbf{g}=sl(2)$ case, with ρ being the fundamental representation, the knowledge of $(\rho\otimes\mathrm{id})\mathcal{F}$ is sufficient to determine the A^i,A^+_i of formulae (2.8) completely. Denoting $n^i=a^ia^+_i$ (with *no* sum over i), $i=\uparrow,\downarrow$ and taking $u=v^{-1}$ one finds for the deformed Weyl algebra

$$\begin{array}{lclclcl} A^+_\uparrow & = & \sqrt{\frac{(n^\uparrow)_{q^2}}{n^\uparrow}}q^{n^\downarrow}a^+_\uparrow & \qquad & A^+_\downarrow & = & \sqrt{\frac{(n^\downarrow)_{q^2}}{n^\downarrow}}a^+_\downarrow \\ A^\uparrow & = & a^\uparrow\sqrt{\frac{(n^\uparrow)_{q^2}}{n^\uparrow}}q^{n^\downarrow} & & A^\downarrow & = & a^\downarrow\sqrt{\frac{(n^\downarrow)_{q^2}}{n^\downarrow}}, \end{array} \tag{2.13}$$

and for the deformed Clifford one

$$\begin{array}{lclclcl} A^+_\uparrow & = & q^{-n^\downarrow}a^+_\uparrow & \qquad & A^+_\downarrow & = & a^+_\downarrow \\ A^\uparrow & = & a^\uparrow q^{-n^\downarrow} & & A^\downarrow & = & a^\downarrow. \end{array} \tag{2.14}$$

In the case that the Hopf algebra H_h is not a genuine quantum group, but a triangular one, the whole discussion simplifies in that one can take trivial u, v, see [9].

Above we have determined in $\mathcal{A}[[h]]$ one particular realization A^i, A^+_j and $\triangleright_h$ of the generators $\tilde{A}^i, \tilde{A}^+_j$ and of the quantum group action. Its main feature is that the $\mathbf{g}$-invariant ground state $|0\rangle$ as well as the first excited states $a^+_i|0\rangle$ of the classical Fock space representation are also respectively $U_h\mathbf{g}$-invariant ground state $|0_h\rangle$ and first excited states $A^+_i|0_h\rangle$ of the deformed Fock space representation.

According to eq. (2.1) all the other realizations are of the form

$$A^{\alpha\, i} = \alpha\, A^i \alpha^{-1} \qquad\qquad A^+_{\alpha\, i} = \alpha\, A^+_i\, \alpha^{-1}, \tag{2.15}$$

with $\alpha = \mathbf{1} + O(h) \in \mathcal{A}[[h]]$. They are manifestly covariant under the realization $\triangleright_{h,\alpha}$ of the $U_h\mathbf{g}$-action defined by

$$x \triangleright_{h,\alpha} a \;:=\; \alpha\, \sigma_h(x_{(\bar{1})})\, a\, \sigma_h(x_{(\bar{2})})\; \alpha^{-1}. \tag{2.16}$$

For these realizations the deformed ground state in the Fock space representation reads $|0_h\rangle = \alpha|0\rangle$; if $\alpha|0\rangle \neq |0\rangle$ the $\mathbf{g}$-invariant ground state and first excited states of the classical Fock space representation do not coincide with their deformed counterparts.

3. CLASSICAL VS. QUANTUM INVARIANTS

We have introduced two actions on $\mathcal{A}[[h]]$:

$$\triangleright : U\mathbf{g} \times \mathcal{A}[[h]] \to \mathcal{A}[[h]], \qquad\qquad \triangleright_h : U_h\mathbf{g} \times \mathcal{A}[[h]] \to \mathcal{A}[[h]]. \tag{3.1}$$

Their respective invariant subalgebras $\mathcal{A}^{inv}[[h]], \mathcal{A}^{inv}_h[[h]]$ are defined by

$$\mathcal{A}^{inv}_h[[h]] := \{ I \in \mathcal{A}[[h]] \mid x \triangleright_h I = \varepsilon_h(x) I \qquad \forall x \in U_h\mathbf{g} \} \tag{3.2}$$

and by the analogous equation where all suffices $_h$ are erased. What is the relation between them? It is easy to prove that [8]

$$\mathcal{A}^{inv}_h[[h]] = \mathcal{A}^{inv}[[h]]. \tag{3.3}$$

In other words invariants under the $\mathbf{g}$-action $\triangleright$ are also $U_h\mathbf{g}$-invariants under $\triangleright_h$, and conversely, although in general $\mathbf{g}$-covariant objects (tensors) and $U_h\mathbf{g}$-covariant ones do not coincide in general!

Let us introduce in the vector space $\mathcal{A}^{inv}[[h]] = \mathcal{A}^{inv}_h[[h]]$ bases $I^1, I^2, ...$ and $I^1_h, I^2_h, ...$ consisting of polynomials respectively in a^i, a^+_j and A^i, A^+_j. It is immediate to realize that we can choose the polynomials homogeneous, since $\triangleright, \triangleright_h$ act linearly without changing their degrees. Explicitly,

$$\begin{array}{ll}
I^1 = a^+_i a^i & I^1_h = A^+_i A^i \\
I^2 = d^{ijk} a^+_i a^+_j a^+_k & I^2_h = D^{ijk} A^+_i A^+_j A^+_k \\
I^3 = d'_{kji} a^i a^j a^k & I^3_h = D'_{kji} A^i A^j A^k \\
I^4 = ... & I^4_h = ...
\end{array} \tag{3.4}$$

where the numerical coefficients $d, d', ...$ form $\mathbf{g}$-isotropic tensors and the numerical coefficients D, D' the corresponding $U_h\mathbf{g}$-isotropic tensors. It is easy to show that $I^1_h \neq I^1$. In general $I^n_h \neq I^n$, although $I^n_h = I^n + O(h)$. The propostion (3.3) implies in particular

$$I^n_h = g^n(\{I^m\}, h) = k^n(\{a^i, a^+_j\}, h). \tag{3.5}$$

What do the 'functions' g^n, k^n, i.e. the formal power series in h with coefficients respectively in $\mathcal{A}^{inv}$ and $\mathcal{A}$, look like?

In Ref. [8] we have found universal formulae yielding the k^n's. The latter turn out to be highly non-polynomial functions, or more precisely in their power expansions in h the degree in a^i, a_j^+ of the corresponding polynomial coefficient grows without bound with the power. It is remarkable that in these universal formulae the twist $\mathcal{F}$ appears only through the coassociator ϕ; therefore all the k^n can be worked out explicitly.

In the case that the Hopf algebra H_h is not a genuine quantum group, but triangular, the coassociator as well as u, v are trivial and one finds $I_h^n = I^n$.

4. FINAL REMARKS, MOTIVATIONS AND CONCLUSIONS

We have shown how one can can realize a deformed $U_h\mathbf{g}$-covariant Weyl or Clifford algebra $\mathcal{A}_h$ within the undeformed one $\mathcal{A}[[h]]$. Given a representation (π, V) of $\mathcal{A}$ on a vector space V, does it provide also a representation of $\mathcal{A}_h$? In other words, can one interpret the elements of $\mathcal{A}_h$ as operators acting on V, if the elements of $\mathcal{A}$ are? If so, which specific role play the elements A^i, A_i^+ of $\mathcal{A}[[h]]$?

In view of the specific example we have examined in ref. [9] the answer to the first question seems to be always positive, whereas the converse statement is wrong: there are more (inequivalent) representations of the deformed algebra than representations of the undeformed ones. This may seem surprising, but is not really a paradox, since the limit $h \to 0$ is smooth for the deforming maps and their inverses only in a h-formal-power sense. Of course, we are especially interested in Hilbert space representations of $*$-algebras. In Ref. [9] we checked that in the operator-norm topology f^{-1} is ill-defined on all but one 'deformed' representation. Roughly speaking, the reason is that the 'particle- number' observables n^i, which enter the transformation f (see e.g. (2.13)) are unbounded operators, therefore even for very small h the effect of the transformation on their large-eigenvalue eigenvectors can be so large to 'push' the latter out of the domain of definition of the operators in $f^{-1}(\mathcal{A})$.

We are especially interested in the case of $*$-algebras admitting Fock space representations. The results presented in the previous paragraphs could in principle be applied to models in quantum field theory or condensed matter physics by choosing representations ρ which are the direct sum of many copies of the same fundamental representation ρ_d; this is what we have addressed in Ref. [14]. The different copies would correspond respectively to different space(time)-points or crystal sites.

One important issue is if $U_h\mathbf{g}$-covariance necessarily implies exotic particle statistics. In view of what we have said the answer is no. At least for compact $\mathbf{g}$ and $U_h\mathbf{g}$ (h is real), the undeformed Fock space representation, which allows a 'Bosons & Fermions' particle interpretation, carries also a representation of the deformed one. Next point is the role of the operators A^i, A_j^+. Quadratic commutation relations of the type (1.6) mean that A_i^+, A^i act as creators and annihilators of some excitations; a glance at (2.8), (2.15) shows that these are not the undeformed excitations, but some 'composite' ones. The last point is: what could the latter be good for. As an Hamiltonian H of the system we can choose a simple combination of the $U_h\mathbf{g}$-invariants I_h^n of section 3; the Hamiltonian is $U_h\mathbf{g}$-invariant and has a simple polynomial structure in the composite operators A^i, A_j^+. H is also $\mathbf{g}$-invariant, but has a highly non-polynomial structure in the undeformed generators a^i, a_j^+ (it would be tempting to understand what kind of physics it could describe!). This suggests that the use of the A^i, A_j^+ instead of the a^i, a_j^+ should simplify the resolution of the corresponding dynamics.

with

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